A NEW FRAMEWORK FOR DYNAMIC CREDIT PORTFOLIO LOSS MODELLING

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Abstract: We present the *SPA framework*, a novel approach to the modeling of the dynamics of portfolio default losses. In this framework, models are specified by a two-layer process. The first layer models the dynamics of portfolio loss distributions in the absence of information about default times. This *background process* can be explicitly calibrated to the full grid of marginal loss distributions as implied by initial CDO tranche values indexed on maturity, as well as to the prices of suitable options. We give sufficient conditions for consistent dynamics. The second layer models the *loss process* itself as a Markov process conditioned on the path taken by the background process. The choice of loss process is non-unique. We present a number of choices, and discuss their advantages and disadvantages. Several concrete model examples are given, and valuation in the new framework is described in detail. Among the specific securities for which algorithms are presented are CDO tranche options and leveraged super-senior tranches.

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1. INTRODUCTION

Credit markets are generally subject to two sources of risk: outright *default risk*, ie, the event risk that an obligor will default on its obligations, and *market risk*, ie, the risk associated with the measures of credit quality (such as credit spreads) widening or narrowing in response to market conditions. In most current models for portfolio credit derivatives, the main focus is on default risk, with little attention paid to the evolution of credit spreads. For instance, in the pricing of options on the portfolio default loss (CDO tranches, etc.), the industry-standard *Gaussian copula* model (see Li (2000)) specifies a distribution of credit events directly and imposes no (sensible) dynamics on the credit spreads. Similar limitations apply to the entire class of factor-based models (see Andersen and Sidenius (2005)), which remains a staple of practical CDO pricing.

Without proper dynamics for credit spreads, a portfolio model lacks the ability to price any kind of instrument that depends on the *dynamic* evolution of loss distributions from their initial values. Such securities include, for instance, options on CDO tranches and forward-starting CDO tranches, both of which are currently traded in the market. With even more complicated securities on tranches, such as Bermuda-style options and interest-rate linked CDO hybrids, not far behind, there is clearly a need for a flexible and tractable

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framework that extends the current portfolio loss models with realistic dynamic properties.

While recent years have seen significant progress made in incorporating the dynamics of credit spreads into single-name credit derivatives models (see, for instance, Schönbucher (2000)), there is, despite the clear appetite from the market, little existing literature on the extension of these models to the to the portfolio case. Apart from the framework described in this paper only Schonbucher (2005) (which we learned of during the process of preparing the present paper[1]) presents a practical approach to incorporating the dynamics of credit spreads into CDO modelling *while simultaneously maintaining exact calibration to CDO markets by construction*. Perhaps the most substantial difference between our model and Schonbucher (2005) lies in the approach taken to the specification of the loss dynamics. In Schonbucher (2005) the starting point is a Markov chain description of the portfolio loss process. The dynamic variables of the model are the forward transition amplitudes of the chain which are initially calibrated to the marginal loss distributions. We, on the other hand, first specify the dynamics of certain "background rates" (in a filtration blinded to defaults) and show that the loss process *conditioned* on the path taken by these rates can be represented as a Markov chain (for which the transition amplitudes can be computed from the conditioning information.) Thus, loosely speaking, both models represent the unconditional loss process as a superposition of Markov chains, but differ on the conditioning and hence on the probabilistic interpretation of this representation. It is too early to say which model is going to be most useful in practice and it is perhaps more fruitful to think of the two approaches as complementary.

Most of the current portfolio loss work, and the current industry standard, focuses attention on the joint distribution of default times—usually parametrized by the choice of some copula function. This approach requires the specification of certain dependence parameters (such as, eg, the correlation of the Gaussian copula) which control the dependence between default times of different obligors, and ultimately results in a model for marginal portfolio loss distributions as observed from time zero. This class of models is, however, essentially static and incapable of specifying evolution of parameters to future times under various market conditions. To introduce dynamics while remaining in calibration to tranche markets (and thereby to marginal loss distributions), we need to consider a mechanism to properly extend the static class of copula models. Let us review a few typical ideas.

One approach to option pricing is to model directly the aggregate portfolio spread, eg, as a jump-diffusion process. The main difficulty here is that this process must reflect co-dependency between both default and credit spreads, and must be flexible enough to allow calibration to the spot CDO market and to leave some freedom for the specification of the volatility of future option prices. In this situation one is in practice faced with the choice between, on the one hand, a relatively parsimonious parameterization which cannot guarantee consistency across option strikes and expirations, but which leads to a manageable model, and on the other hand a model which can capture the whole market at the price of unstable parameters and, very likely, unrealistic forward dynamics.

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[1] We gratefully acknowledge our fruitful discussions with Philipp Schönbucher.

Another possibility for a dynamic model is to model the credit spreads of individual obligors as distinct jump-diffusion processes while introducing some explicit mechanism (eg, *frailty variables*, *simultaneous spread shocks*) for generating co-dependencies of spreads and defaults. The main issue here is the large number of free parameters involved and the attendant difficulties with calibration and numerical computations. Although theoretically interesting and well-covered in the literature (see, eg, Duffie and Singleton (1999) and Schonbucher (2003)) such models seem at least at present to be of little practical use.

In this paper we develop the *SPA model*, a framework in which, loosely speaking, the loss distributions themselves are the primary state variables. It is clear that the CDO tranche prices across expiries/detachment levels as observed at $t = 0$ depend on marginal distributions of the loss process only, and do not contain any dynamic information (or, equivalently, are consistent with a wide range of possible dynamics). By making the loss distribution itself a dynamic variable, and ensuring that it starts at the current "value"—which in practice will have to be constructed from available market quotes[2]—a tractable framework is built. The approach can be seen to extend the HJM approach in interest rate modeling, with the portfolio loss distribution in our setup playing the role of the term structure of zero-coupon bonds in the HJM approach. More specifically, for every value of fractional portfolio loss and for every time we consider the implied initial probability that this fractional loss will not be exceeded at the considered time. For a fixed loss fraction we get in this way an initial term structure of (analogs to) "zero-coupon bond prices" which we can describe in terms of an initial forward "market" rate curve. By propagating the initial curve to future times we generate future term structures of loss probabilities (and, thus, tranche spreads). We assume a diffusion process for the forward rates, requiring their dynamics to be specified under a filtration which does not contain information about default times. However, in order to facilitate pricing we need to be able to condition on realized future losses and we shall consequently have to introduce a larger filtration which includes information about defaults. We refer to the small filtration without defaults as the *background* filtration and to the larger filtration as the *loss* filtration.

The intuition for our set-up is that the intensity of a loss at any given time depends on certain market information (including, for example implied default co-dependency and the general state of the market) as well as the actual losses up to that time. Conditioned on the background information, ie, on the path taken by the forward rates, a loss process may be constructed which is consistent with the conditioning loss distributions.

For each loss fraction and time the corresponding forward rate is easily shown to be a martingale under a suitably defined forward measure and so the dynamics of the (one-parameter family of) forward rate structures is completely specified by the volatility structure. The interpretation in terms of probability distributions imposes certain constraints on volatilities, but still a very rich specification is possible. We give a simple, but non-trivial, example of a consistent forward rate dynamics.

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[2] Note that this is a non-trivial step which requires some further modeling choices (which we shall not discuss in the present paper.) For example, interpolation in maturity as well as loss level will be required since CDO tranches are quoted in the market only for a few maturities and detachment levels. Furthermore, since CDO tranches are quoted on *spreads* rather than *present values*, the risk-neutral expectation of the loss on a tranche cannot be implied directly from the market.

The framework presented in this paper can be seen as a generalization of the single-obligor model of Schönbucher (2000) to the portfolio case. In particular, in the case of a portfolio consisting of single obligor, the SPA model reduces to the well-known Cox-process-based model for the stochastic credit spread.

The rest of this paper is organized as follows. Section 2 defines the model set-up and Section 3 presents the dynamics and discusses the consistency requirements. In Section 4 we present a number of explicit constructions of a loss process consistent with the dynamics of the loss distribution, and discuss their advantages and disadvantages. In Section 5 we give detailed algorithms for Monte-Carlo valuation of simple and more complex securities dependent on the dynamics of the portfolio loss. Section 6 presents simple models following the SPA framework. Section 7 concludes the paper.

2. MODEL SET-UP

The SPA model takes as its state variable the portfolio loss distribution and we therefore formulate dynamics in terms of transition probabilities. We construct the model "in reverse", ie, we first construct the dynamics of loss distributions and then construct a process for losses conditional on the path taken by the loss distributions. A slightly subtle point concerns the information available at a given time through the relevant filtration. When we construct the dynamics of loss *distributions* we work with the background filtration which, crucially, does not contain information about default times. This allows the dynamics to be smooth, even when the loss process itself has jumps; loosely speaking we are smoothing the dynamics of distributions by not observing ("integrating out") losses. We shall subsequently construct a loss process, ie, the full collection of *conditional* loss distributions, consistent with all information revealed by the background filtration[3] (over some time interval $[0,T_{\max}]$, where $T_{\max}$ is the model horizon.) Naturally, in order to *sample* the loss process we need to work with the larger loss filtration which reveals all information about losses.

For simplicity we assume vanishing, deterministic, risk-free interest rates[4]. Consider a portfolio of credit-risky securities (eg bonds or credit default swaps), and let the time t portfolio notional be $A(t)$ with initial notional $A(0)$. (The initial notional $A(0)$ can be less than the total notional of the pool to allow for credit losses prior to $t=0$.) Clearly, we have

$$A(T)\geq A(T'),\ T\leq T'. \tag{1}$$

We introduce the *loss fraction*

$$l(T):=1-A(T)/A(0),$$

which has initial value $l(0)=0$ and, by (1), must be non-decreasing:

$$l(T)\leq l(T'),\ T\leq T'. \tag{2}$$

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[3] This requires additional modeling assumptions since the process is not uniquely determined by the unconditional distributions.

[4] The extension to the case of stochastic risk-free rates independent of the loss dynamics is straightforward. The introduction of non-trivial coupling between interest rates and default rates is also in principle straightforward, but requires some work. We do not consider this problem here.

Assumption 2.1. *Formally, we restrict the dynamics of the loss process to the following class. Let the probability space be equipped with the filtration* $\{\mathscr{L}_t\}$. *It is assumed that there exists a proper sub-filtration* $\{\mathscr{M}_t\}$, *with* $\mathscr{M}_0 = \mathscr{L}_0$, *such that when conditioned on* $\{\mathscr{M}_{T_{\max}}\}$ *the loss process* $l(t)$ *is Markovian. In particular, the parameters of the conditional Markov process* $l(t)$ *(such as transition probabilities, etc) are assumed to be measurable with respect to the background filtration* $\{\mathscr{M}_t\}$.

Define for any $x \in [0,1]$ the stopping time τ_x as the random variable defined by

$$\tau_x := \inf\{t \geq 0 | l(t) > x\}, \tag{3}$$

ie, τ_x is a first crossing time. We can think of the crossing time as the first jump time of a Cox process with non-negative (stochastic) intensity $\lambda_x(t)$.Now let P be the risk-neutral probability measure and E the associated expectation operator. For all $x \in [0,1]$ we define the quantities

$$\begin{aligned} p_x(t,T) := {} & \mathsf{P}\left(\tau_x > T | \mathscr{M}_t\right) \\ = {} & \mathsf{P}\left(l(T) \leq x | \mathscr{M}_t\right) \\ = {} & \mathsf{E}\left(e^{-\int_0^T \lambda_x(s)ds} \middle| \mathscr{M}_t\right) \\ = {} & e^{-\int_0^t \lambda_x(s)ds} \mathsf{E}\left(e^{-\int_t^T \lambda_x(s)ds} \middle| \mathscr{M}_t\right). \end{aligned} \tag{4}$$

We assume that $p_x(0,T)$ is observable at time 0 for all T, eg from observation of market prices of tranche securities (see Remark 5.1). Evidently, we must have $p_x(0,0) = 1$ and

$$p_1(t,T) \equiv 1, \tag{5a}$$

$$p_x(t,T) \geq p_x(t,T'),\ T \leq T', \tag{5b}$$

$$p_x(t,T) \leq p_y(t,T),\ x \leq y. \tag{5c}$$

Note that condition (5b) requires $p_x(t,T)$ to be decreasing in maturity, whilst (5c) requires it to be increasing in loss fraction. In order to ensure consistency we must demonstrate that if (5) hold for $t = 0$, they will hold also for all $t > 0$, ie, the stochastic dynamics of the p's will not generate states violating these conditions. We shall return to this issue in Section 3 below. As an aside, note that we allow for $t > T$ in the definition of $p_x(t,T)$, in which case we have

$$p_x(t,T) = e^{-\int_0^t \lambda_x(s)ds} e^{-\int_t^T \lambda_x(s)ds} = p_x(T,T), \quad t > T. \tag{6}$$

For future reference define

$$\begin{aligned} \pi_x(t,T) \ := {} & \frac{\partial}{\partial x} p_x(t,T) \\ = {} & \mathsf{P}\left(l(T) \in [x, x+dx] | \mathscr{M}_t\right)/dx. \end{aligned} \tag{7}$$

Define time-t *continuously compounded forward rates* by

$$f_x(t,T) := -\frac{\frac{\partial}{\partial T} p_x(t,T)}{p_x(t,T)}, \tag{8}$$

ie,

$$p_x(t,T) = e^{-\int_0^t f_x(u,u)du} e^{-\int_t^T f_x(t,u)du},$$

where

$$f_x(t,t) \equiv \lambda_x(t).$$

From (6) it follows that $f_x(t,T) = f_x(T,T)$ for $t > T$. From (8) we see that

$$p_x(t,T') \equiv p_x(t,T)e^{-\int_T^{T'} f_x(t,u)du}, \quad T \leq T'. \tag{9}$$

For later use, we emphasize that for each x, the quantity $p_x(t,T)$ is a martingale in the background filtration. Specifically, for $t' > t$, by iterated expectations

$$\mathsf{E}\left(p_x(t',T)\middle|\mathscr{M}_t\right) = \mathsf{E}\left(\mathsf{E}\left(1_{\{\tau_x>T\}}\middle|\mathscr{M}_{t'}\right)\middle|\mathscr{M}_t\right) = \mathsf{E}\left(1_{\{\tau_x>T\}}\middle|\mathscr{M}_t\right) = p_x(t,T).$$

On the other hand, note that

$$p_x(t,t) = e^{-\int_0^t \lambda_x(s)ds}$$

such that $dp_x(t,t)/p_x(t,t) = -\lambda_x(t)dt$ (making $p_x(t,t)$ a predictable supermartingale).

3. LOSS DISTRIBUTION DYNAMICS

In the background filtration, the loss distribution at time t can be described either in terms of the probabilities $p_x(t,T)$ or, equivalently, in terms of forward rates $f_x(t,T)$. Hence, the dynamics of the loss distribution can also be specified in terms of either of these quantities. It will become clear later that the approach we are proposing follows the so-called Heath-Jarrow-Morton, or HJM, approach to constructing interest rate term structure models (see Heath et al. (1992)). In the HJM framework, the dynamics are initially specified in terms of the forward rates (ie, counterparts to $f_x(\cdot,\cdot)$) but, as is well known, can be equivalently specified in terms of the dynamics of $p_x(\cdot,\cdot)$ instead. In fact, we find the ability to switch between these equivalent views convenient for formulating the conditions that a model must satisfy, and will use that below.

3.1. **Loss probability models.** We have demonstrated in the previous chapter that for each x, T, the process for $p_x(\cdot,T)$ must be a (positive) martingale in the background filtration. Thus, restricting ourselves to the diffusion framework (which seems quite sensible as, by definition, $p_x(\cdot,T)$ is a process for the portfolio loss that can evolve continuously even if the loss process itself has jumps), we can write, most generally,

$$dp_x(t,T)/p_x(t,T) = \Sigma_x(t,T)\,dW_x(t), \tag{10}$$

with the requirement that

$$\Sigma_x(T,T) = 0.$$

Here $\Sigma_x(t,T)$ is a general stochastic process (in t) indexed by x, T, and $W_x(t)$ is a Brownian motion for each x with respect to the background filtration.

Under suitable differentiability assumptions, the dynamics for the forward rates follow by Ito's lemma,

$$df_x(t,T) = \sigma_x(t,T)\Sigma_x(t,T)\,dt + \sigma_x(t,T)\,dW_x(t), \tag{11}$$

where

$$\sigma_x(t,T) = -\frac{\partial}{\partial T}\Sigma_x(t,T), \quad \Sigma_x(t,T) = -\int_t^T \sigma_x(t,u)\,du. \tag{12}$$

The condition (5a) is ensured if the volatility process $\Sigma_x(t,T)$ is chosen so that

$$\Sigma_1(t,T) \equiv 0. \tag{13}$$

The condition (5b) is similar to the requirement of non-decreasing term structure of zero-coupon discount bonds in the HJM framework. The necessary (and sufficient) condition for that is non-negativity of the forward rate processes defined by (11),

$$f_x(t,T) \geq 0 \quad \text{for all } t,T,x.$$

Hence, as far as the condition (5b) is concerned, imposing the dynamics on forward rates is more natural.

The condition (5c) does not have an equivalent in the standard HJM approach, and is unique to our setup. Moreover, this condition is in fact quite strong, imposing significant restrictions on the allowed dynamics. This is most clearly seen in an extreme example of $W_x(t)$'s being all independent for different x. If this is the case (and, for simplicity, $\Sigma_x(t,T)$ are all the same and constant, $\Sigma_x(t,T) \equiv \Sigma$), then the trajectories of $p_{x_1}(t,T)$ and $p_{x_2}(t,T)$ for $x_1 \neq x_2$ will cross with positive probability, and the condition (5c) will be violated.

In order to satisfy the condition (5c), a strong *spatial* (ie, in x direction) dependence on the driving Brownian motions need to be imposed. It is also intuitively clear which type of condition is needed. We shall require that, as $p_{x_1}(t,T)$ and $p_{x_2}(t,T)$ get closer, the diffusion coefficient of their difference becomes smaller and smaller, so that the point 0 is an absorbing state for the difference $p_{x_1}(t,T) - p_{x_2}(t,T)$ (for each x_1, x_2).

The same condition expressed in terms of f's would be more complicated and would involve not only the diffusion coefficients but the drifts as well. We shall investigate this approach later in the paper, see Theorem 3.2. First, however, we concentrate on the necessary condition of "spatial ordering preservation" in terms of the dynamics imposed on the p's.

It should be clear from the discussion so far that all dW_x's must be strongly linked. One way to achieve this is to require that $\{W_x(t), x \in [0,1], t \geq 0\}$ is a *Brownian sheet* with certain regularity conditions in x, not unlike the approach taken to define the so-called *stochastic string shock* models of interest rates (see Santa-Clara and Sornette (2001)). To stay in a context familiar to most researchers in the area of quantitative finance (and, importantly, to avoid obscuring key ideas with unnecessary details) we consider an almost equivalent framework with a potentially infinite number of standard Brownian motions driving the processes for p_x for all x. Since, ultimately, in practical applications only a finite number of "levels" x_j will be involved in calculations, the flexibility of the framework is not restricted.

Let us consider a (possibly infinite) collection of independent Brownian motions $dW^\alpha(t)$, $\alpha \geq 1$, and a collection of stochastic processes $\Sigma_x^\alpha(t,T)$, $\alpha \geq 1$, for all x,t,T. The dynamics of all $p_x(\cdot,\cdot)$ are defined by

$$dp_x(t,T)/p_x(t,T) = \sum_\alpha \Sigma_x^\alpha(t,T)\, dW^\alpha(t), \tag{14}$$

and the dynamics of $f_x(\cdot,\cdot)$ by

$$df_x(t,T) = \sum_\alpha \sigma_x^\alpha(t,T)\Sigma_x^\alpha(t,T)\, dt + \sum_\alpha \sigma_x^\alpha(t,T)\, dW^\alpha(t), \tag{15}$$

where

$$\sigma_x^\alpha(t,T) = -\frac{\partial}{\partial T}\Sigma_x^\alpha(t,T)$$

or, equivalently,

$$\Sigma_x^\alpha(t,T) = -\int_t^T \sigma_x^\alpha(t,u)\,du. \tag{16}$$

Assume that the processes $\{\sigma_x^\alpha(\cdot,T)\}$ satisfy all the necessary conditions for $f_x(t,T)$ (and $p_x(t,T)$) to be well-defined by these equations. Rigorous treatment can proceed along the same lines as in Miltersen (1994) for each fixed x. For our purposes it is enough to require that

- For each α, x, T,

$$\{\sigma_x^\alpha(\cdot,T)\} \quad \text{is an } \{\mathscr{M}_t\}\text{-adapted process;} \tag{17}$$

- The following holds

$$\int_0^T \left|\sum_\alpha \sigma_x^\alpha(t,T)\Sigma_x^\alpha(t,T)\right| dt + \int_0^T \sum_\alpha \left(\sigma_x^\alpha(t,T)\right)^2 dt < \infty \tag{18}$$

P-almost surely.

The following theorem is the main result on the conditions sufficient for (5c) to hold, expressed in terms of the process for loss probabilities $p_x(t,T)$.

Theorem 3.1. *Suppose the condition (5c) is satisfied for $t=0$, a family $\{\sigma_x^\alpha(\cdot,T)\}$ satisfying (16), (17), (18) is given, and the dynamics of $p_x(\cdot,T)$ and $f_x(\cdot,T)$ are given by (14), (15). If for any x,y,T and $\varepsilon > 0$ there exists a constant C (that may depend on ε, x, y, T) such that for all $t > 0$*

$$\sum_\alpha \left(p_x(t,T)\Sigma_x^\alpha(t,T) - p_y(t,T)\Sigma_y^\alpha(t,T)\right)^2 \le C\left|p_x(t,T) - p_y(t,T)\right| \tag{19}$$
$$\text{for } \left|p_x(t,T) - p_y(t,T)\right| < \varepsilon,$$

then (5c) holds for any $t \ge 0$.

Proof. See Appendix A.
■

Remark 3.1. *The condition (19) is a flavor of the Yamada condition for the existence of strong solutions, see Yamada and Watanabe (1971) or (Karatzas and Shreve, 1991, Proposition 2.13).*

Remark 3.2. *As is clear from the proof, a weaker condition than (19) is sufficient,*

$$\mathsf{E}\left(\sum_\alpha \left(p_x(t,T)\Sigma_x^\alpha(t,T) - p_y(t,T)\Sigma_y^\alpha(t,T)\right)^2 \middle| p_x(t,T) - p_y(t,T)\right) \le C\left|p_x(t,T) - p_y(t,T)\right|$$
$$\text{for } \left|p_x(t,T) - p_y(t,T)\right| < \varepsilon.$$

This condition may be harder to check, however.

While the condition (19) is very general, it could be difficult to check for general volatility processes. The easiest way to ensure that it holds is to require that the volatility processes $\Sigma_x^\alpha(t,T)$ are deterministic functions of p's. It seems natural to require that each $\Sigma_x^\alpha(t,T)$ is a function of $p_x(t,\cdot)$ only (and not of $p_y(t,\cdot)$ for $y \neq x$). On the other hand, from the large body of work on HJM models available, it is clear that requiring $\Sigma_x^\alpha(t,T)$ to be a deterministic function of $p_x(t,T)$ only (and not of $p_x(t,u)$ for $u \neq T$) is too restrictive. Hence, it would be interesting to look for functions $\Sigma_x^\alpha(t,T)$ that satisfy (19) in the class of

$$\Sigma_x^\alpha(t,T) = \phi^\alpha\left(t, \{p_x(t,s)\}_{s \geq T}\right),$$

where ϕ^α are deterministic functions.

Mainly for notational convenience, assume we can find a set of "offsets" $0 = \delta_1, \delta_2, \ldots, \delta_K$ such that for any x, α,

$$\Sigma_x^\alpha(t,T) = \phi^\alpha\left(t, \{p_x(t, T + \delta_k)\}_{k=1}^K\right), \tag{20}$$

where $\phi^\alpha(t,\bar{p})$, $\bar{p} = (p_1, \ldots, p_K)$, is

$$\phi^\alpha(t,\cdot) : \mathbb{R}^K \to \mathbb{R}.$$

Then the condition (19) can be rewritten as

$$\sum_\alpha (p_1 \phi^\alpha(t,\bar{p}) - q_1 \phi^\alpha(t,\bar{q}))^2 \leq C\,|p_1 - q_1|, \tag{21}$$

where $\bar{p} = (p_1, \ldots, p_K) \in \mathbb{R}^K$, $\bar{q} = (q_1, \ldots, q_K) \in \mathbb{R}^K$. This can be seen as a condition similar to, although not quite the same as, the one normally required for the existence of strong solutions, ie, the Lipschitz continuity condition

$$\sum_\alpha (p_1 \phi^\alpha(\bar{p}) - q_1 \phi^\alpha(\bar{q}))^2 \leq C_2 \|\bar{p} - \bar{q}\|^2.$$

3.2. **Instantaneous forward rate models.**

Having derived the sufficient "spatial order preservation" conditions in terms of the dynamics of loss probabilities, we turn our attention to the dynamics of forward rates $f_x(t,T)$. Next, we shall obtain an alternative to Theorem 3.1 with conditions imposed on forward rate volatilities $\sigma_x(t,T)$.

The necessary no-arbitrage conditions on the loss distribution dynamics can be recast *equivalently* in forward rate terms,

$$f_1(t,T) \equiv 0 \tag{22a}$$

$$f_x(t,T) \geq 0, \tag{22b}$$

$$\int_0^t f_x(u,u)du + \int_t^T f_x(t,u)du \geq \int_0^t f_y(u,u)du + \int_t^T f_y(t,u)du,\ x \leq y. \tag{22c}$$

Here (22a) is a simple "saturation boundary" while (22b) is implied by unitarity (probabilities cannot be greater than one). The third condition is by far the most constraining. A sufficient, but clearly not necessary, condition for (22c) to hold is

$$f_x(t,T) \leq f_y(t,T) \text{ everywhere},\ 0 \leq y \leq x \leq 1,\ 0 \leq t \leq T. \tag{23}$$

Theorem 3.2. *Suppose the conditions (22a), (22b) and (23) are satisfied for* $t = 0$, *a family* $\{\sigma_x^\alpha(\cdot, T)\}$ *satisfying (16), (17), (18) is given, and the dynamics of* $p_x(\cdot, T)$ *and* $f_x(\cdot, T)$ *are given by (14), (15). If the volatility processes* $\{\sigma_x^\alpha(\cdot, T)\}$ *can be represented in the form*

$$\sigma_x^\alpha(t,T) = \varphi^\alpha(t,T,f_x(t,T))$$

where each $\varphi^\alpha(t,T,\cdot)$ *is a continuous function* $\mathbb{R} \to \mathbb{R}$ *such that*

$$v_1 \le v_2 \Rightarrow |\varphi^\alpha(t,T,v_1)| \le |\varphi^\alpha(t,T,v_2)| \text{ for any } \alpha, t, T, v_1, v_2$$

and

$$\varphi^\alpha(t,T,0) \equiv 0,$$

then (22a), (22b) and (23) (and, therefore, (22c) hold for any $t \ge 0$.

Proof. See Appendix B.
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3.3. **Forward Libor models.** In recent years, another parametrization of interest rate dynamics, often called *forward Libor models*, has gained a certain degree of popularity. In this approach, forward Libor rates are taken as primitives of the model, and the dynamics of the interest rate curve are expressed directly in terms of them. The SPA framework can be reformulated in terms of (simple) forward rates as well.

Assume a tenor structure

$$\begin{aligned} &T_0 < T_1 < \cdots < T_N, \\ &\Delta_n = T_n - T_{n-1}, \end{aligned}$$

is given, and define spanning forward Libor rates by

$$F_x(t,n) = \frac{p_x(t,T_n) - p_x(t,T_{n+1})}{\Delta_n p_x(t,T_{n+1})}, \quad n = 0, \ldots, N-1. \tag{24}$$

It follows from (14) that for $t < T_n$,

$$\begin{aligned} dF_x(t,n) &= \Delta_n^{-1} d\left(\frac{p_x(t,T_n)}{p_x(t,T_{n+1})}\right) \\ &= \Delta_n^{-1}\frac{p_x(t,T_n)}{p_x(t,T_{n+1})}\left(\sum_\alpha \left(\Sigma_x^\alpha(t,T_n) - \Sigma_x^\alpha(t,T_{n+1})\right) dW^\alpha(t)\right) \\ &+ \Delta_n^{-1}\frac{p_x(t,T_n)}{p_x(t,T_{n+1})}\left(\sum_\alpha \left((\Sigma_x^\alpha(t,T_n))^2 - (\Sigma_x^\alpha(t,T_{n+1}))^2\right)\right) dt \\ &= \frac{1+\Delta_n F_x(t,n)}{\Delta_n}\sum_\alpha \left(\Sigma_x^\alpha(t,T_n) - \Sigma_x^\alpha(t,T_{n+1})\right) \\ &\times \left(dW^\alpha(t) + \left(\Sigma_x^\alpha(t,T_n) + \Sigma_x^\alpha(t,T_{n+1})\right) dt\right). \end{aligned}$$

Let us choose $\Sigma_x^\alpha(t,T_n)$'s to be of a particular form. We fix a deterministic local volatility function $\psi(x)$ and a collection of deterministic time-dependent volatilities $\gamma^\alpha(t,n)$ (same for all x) and define

$$\begin{aligned}(25)\qquad &\Sigma_x^\alpha(t,T_{n+1}) = \Sigma_x^\alpha(t,T_n) + \gamma^\alpha(t,n)\frac{\Delta_n\psi(F_x(t,n))}{1+\Delta_n F_x(t,n)}, \quad n = n(t),\dots,N-1,\\ &\Sigma_x^\alpha\left(t,T_{n(t)}\right) = 0,\end{aligned}$$

where we have defined $n(t)$ by the condition $T_{n(t)-1} \le t < T_{n(t)}$. Then

$$\begin{aligned}(26)\qquad dF_x(t,n) &= \psi(F_x(t,n))\sum_\alpha \gamma^\alpha(t,n)\left(dW^\alpha(t) + \mu_x^\alpha(t,n)\,dt\right),\\ \mu_x^\alpha(t,n) &= \sum_{k=n(t)}^{n}\frac{\Delta_k\psi(F_x(t,k))}{1+\Delta_k F_x(t,k)}\gamma^\alpha(t,k).\end{aligned}$$

Following Brace et al. (1996), we can show that a consistent model can indeed be defined by (25) and (26). The model is, in fact, an extension of the *skew-extended forward Libor model* of interest rates (see Andersen and Andreasen (2000)) to loss probabilities modelling.

Let $\mathsf{P}^{x,T_{n+1}}$ be the probability measure under which

$$dW^{\alpha,x,T_{n+1}}(t) \triangleq dW^\alpha(t) + \mu_x^\alpha(t,n)\,dt, \quad \alpha \ge 1,$$

is a (multi-dimensional) driftless Brownian motion. Under this measure $F_x(\cdot,n)$ is a martingale, and

$$(27)\qquad dF_x(t,n) = \psi(F_x(t,n))\sum_\alpha \gamma^\alpha(t,n)\,dW^{\alpha,x,T_{n+1}}(t).$$

It is not hard to see that $\mathsf{P}^{x,T_{n+1}}$ corresponds to $p_x(\cdot,T_{n+1})$ being used as a numeraire.

The volatility structure in the model (26) is specified by the local volatility function[5] $\psi(x)$ and the volatility structure $\{\gamma^\alpha(t,n)\}_{\alpha,n}$. The necessary conditions for satisfying (5) can be expressed using such a parametrization as well. The discussion follows closely the arguments from Section 3.2 on the model expressed in terms of the (instantaneous) forward rates. To avoid repetition, we only sketch out the arguments.

To satisfy (5a) and (5b), the point 0 needs to be absorbing for $F_x(\cdot,n)$ for all x, n. Considering the process $F_x(\cdot,n)$ under its natural measure as in (27), we see that this holds if the Yamada condition on $\psi(x)$ holds, roughly $|\psi(x)| \le |x|^{1/2}$ for small $|x|$. In particular, it is required that $\psi(0) = 0$.

We require that

$$F_1(t,n) = 0 \text{ for all } t,n.$$

To make sure that (5c) is satisfied for any $t \ge 0$, we shall also require that

$$(28)\qquad F_x(0,n) \le F_y(0,n) \text{ for } 0 \le y \le x \le 1 \text{ and any } n = 0,\dots,N-1,$$

and

$$(29)\qquad \frac{\psi(v_1)}{1+\Delta_k v_1} \le \frac{\psi(v_2)}{1+\Delta_k v_2} \text{ for } v_1 \le v_2 \text{ and any } k = 0,\dots,N-1.$$

---

[5] Note that the function can be chosen to depend on n and t as well; this trivial extension is not considered in this paper to preserve the clarity of exposition. Stochastic volatility extensions are also possible.

Analyzing the drift of $F_y(t,n) - F_x(t,n)$ at the time τ, the first time for which there exists n such that $F_y(t,n) - F_x(t,n) = 0$, we see that these two conditions are sufficient to guarantee that $F_x(t,n) \leq F_y(t,n)$ holds for any $t \geq 0$, any $0 \leq y \leq x \leq 1$ and any $n = 0, \ldots, N-1$. This in turn is sufficient for (5c).

4. THE LOSS PROCESS

In general, a portfolio loss process is equivalent to the collection of all conditional probabilities $\mathsf{P}(l(t) \leq x | \mathscr{L}_s)$ for all x and all s, t with $s < t$. Given Assumption 2.1, a loss process *conditioned on the path taken by the forward rates* over the interval $[0,t]$, ie, conditioned on $\mathscr{M}_t$, must reproduce the loss probabilities (4). It is not clear whether such a loss process exists (in the category of conditional Markov processes) and whether it is unique. Furthermore, the question of the existence of such a loss process with otherwise "natural" properties is crucial, because many pricing problems involve an explicit dependence on losses.

4.1. **Loss process as a one-step Markov chain.**

To demonstrate existence under very general conditions, and to provide a useful, generic loss process we shall now show that, conditioned on the path taken by the forward rates, the loss process may be constructed as a simple one-step Markov chain. As we shall see, the transition probabilities of this process for some time t will be uniquely determined by the background forward rates for that time; loosely speaking the state of the market determines the intensity of the loss process. Before commencing on our construction, it may be noted that Markov chains have long been used in default modeling (see eg Jarrow et al. (1997) for the single obligor case and Graziano and Rogers (2005) for an example of portfolio modeling), but in most past work the states of the chain represent the credit quality of the individual firms rather than, as here, states of an aggregate portfolio loss process.

We first consider some fixed time t and let the loss probabilities $p_x(t,T)$ be given for all x, T. We have for general T,

$$\begin{aligned}
\mathsf{P}(\tau_x \in [T, T+dT] | \mathscr{M}_t) &= \mathsf{P}(\tau_x > T | \mathscr{M}_t) - \mathsf{P}(\tau_x > T + dT | \mathscr{M}_t) \\
&= p_x(t,T) - p_x(t, T+dT) \\
&= -\frac{\partial}{\partial T} p_x(t,T)\, dT
\end{aligned}$$

Next, assume that loss fractions can take values only on a finite grid[6] $0 = x_0 < x_1 < \ldots < x_N = 1$ and further assume that the loss can shift up by at most one step at a time, ie, if $l(T) = x_i$, then at time T the loss can only change by jumping to x_{i+1}. Let $\kappa_{x_i}(t,T)$ be the intensity conditioned on $\mathscr{M}_t$ for a jump from x_i to x_{i+1} at time T, ie,

$$\kappa_{x_i}(t,T)\, dT = \mathsf{P}(l(T+dT) = x_{i+1} | l(T) = x_i, \mathscr{M}_t).$$

---

[6] The size of the steps in the x-grid may reasonably be tied to the loss-given-defaults for the firms in the portfolio, but we shall leave the discussion of the choice of loss discretization for future research.

Then

$$\begin{aligned}\mathsf{P}\left(\tau_{x_i}\in[T,T+dT]\middle|\mathscr{M}_t\right) &= \mathsf{P}\left(l(T+dT)=x_{i+1},l(T)=x_i\middle|\mathscr{M}_t\right)\\ &= \mathsf{P}\left(l(T+dT)=x_{i+1}\middle|\mathscr{M}_t,l(T)=x_i\right)\mathsf{P}\left(l(T)=x_i\middle|\mathscr{M}_t\right)\\ &= \kappa_{x_i}(t,T)\,\mathsf{P}\left(l(T)=x_i\middle|\mathscr{M}_t\right)dT.\end{aligned}$$

Putting everything together we see that a sufficient requirement is

$$-\frac{\partial}{\partial T}p_x(t,T)=\kappa_{x_i}(t,T)\,\mathsf{P}\left(l(T)=x_i\middle|\mathscr{M}_t\right).$$

For $\mathsf{P}\left(l(T)=x_i\middle|\mathscr{M}_t\right)>0$ we can solve uniquely for the intensity. In fact,

$$\kappa_{x_i}(t,T)=\frac{-\frac{\partial}{\partial T}p_{x_i}(t,T)}{\mathsf{P}\left(l(T)=x_i\middle|\mathscr{M}_t\right)},$$

where

$$\mathsf{P}\left(l(T)=x_i\middle|\mathscr{M}_t\right)=\begin{cases}p_0(t,T), & i=0\\ p_{x_i}(t,T)-p_{x_{i-1}}(t,T), & i>0\end{cases}.$$

Furthermore, from (5b) and (5c) follows that all of these intensities are non-negative. For $\mathsf{P}\left(l(T)=x_i\middle|\mathscr{M}_t\right)=0$ any value for the intensity will satisfy the requirement. By explicitly constructing a Markov chain, we have proven the following theorem.

Theorem 4.1. *Suppose the conditions (5) hold. Then, for any discretization $0=x_0<x_1<\ldots<x_N=1$ for which $p_{x_{i+1}}(t,T)>p_{x_i}(t,T)$, a non-decreasing, one-step loss process $l(\cdot)$, $l(0)=0$, on the state space $\{x_j\}_{j=0}^N$ can be found such that it is consistent with the processes for the loss probabilities $\{p_x(t,T)\}$ for $x\in\{x_j\}_{j=0}^N$. In particular, for any x,t,T , $x\in\{x_j\}_{j=0}^N$,*

$$\mathsf{P}\left(l(T)\leq x\middle|\mathscr{M}_t\right)=p_x(t,T).$$

Remark 4.1. *The loss process $l(s)$ is constructed for $s\in[0,t]$, where t is the time to which the loss distributions are propagated (as evidenced by the subscript t in the background filtration $\mathscr{M}_t$). The time t should be chosen as the final date to which the loss process needs to be constructed in order to value a given security. We will give suitable examples later in the paper.*

Remark 4.2. *For $s\leq t$ we have*

$$\kappa_{x_i}(t,s)=\kappa_{x_i}(s,s).$$

Hence, to propagate the loss process $s\longrightarrow s+ds$, the loss distributions at time s, instead of at time t, can be used. This can be beneficial in Monte-Carlo simulations, and we shall return to this in later chapters.

4.2. **Loss process as a general Markov chain.** The algorithm from Section 4.1 provides an explicit construction of the loss process that is defined on a state space $\{x_j\}_{j=0}^N$ which is a subset of $[0,1]$. One of the drawbacks of this construction is that it is only consistent with the portfolio loss distributions $p_x(\cdot,\cdot)$ for $x\in\{x_j\}_{j=0}^N$. A question remains, whether a process consistent with the loss distributions *for all* $x\in[0,1]$ can be constructed.

The utility of such a construction can be questioned, as one can argue that for a homogeneous portfolio, loss-to-date would always be in the multiples of LGD (loss-given-default) per name. This argument, however, does not take into account that fact that we might wish to consider the case of stochastic LGDs, in which case the range of possible losses may include the whole interval $[0,1]$. Moreover, even in the situation of non-stochastic but idiosyncratic LGDs, the “unit of loss”, ie an amount such that any LGD is an integer multiple of it, can be very small and thus, the discretization very fine.

One way to approach the problem is to take a limit of one-step Markov chains as the discretization of the state space is refined. As shown in Appendix C, the resulting process exhibits rather questionable dynamics, indicating that not only the continuous-space limit is inappropriate as a model for the loss process, but also that for finely-discretized state space, the one-step Markov chain is questionable.

In view of the limitations of representing the loss process as a simple Markov chain, let us consider loss processes of more general form. Clearly, strong assumptions on the type of the loss process still need to be made to ensure that the (weak) constraints on the marginal one-dimensional distributions specify the loss process uniquely within the selected class. The process will still be constructed in the class of conditional Markovian processes per Assumption 2.1. What will be relaxed is the requirement that the loss process jumps to the nearest neighbor only.

Recall that our goal is to construct a conditional Markov process for the losses $l(\cdot)$ that is consistent with the family of *a-priori* given loss probabilities

$$p_x(t,T) = \mathsf{P}\left(l(T) \le x \middle| \mathscr{M}_t\right).$$

Definition 4.1. *The* jump survival function $m_{z,x}(t,T)$ *is defined by*

$$m_{z,x}(t,T) = \mathsf{P}\left(l(T+dT) > x \middle| l(T) = z, \mathscr{M}_t\right) \big/ dT.$$

The jump density $\mu_{z,x}(t,T)$ *is defined by*

$$\mu_{z,x}(t,T) = \frac{\partial}{\partial x} m_{z,x}(t,T) = \mathsf{P}\left(l(T+dT) \in [x, x+dx] \middle| l(T) = z, \mathscr{M}_t\right) \big/ (dT\,dx).$$

Proposition 4.1. *If $l(t)$ be a non-decreasing pure-jump conditional Markov process on the state space $[0,1]$, then*

$$m_{z,1}(t,T) = 0 \text{ for all } t, T, z, \tag{30}$$

and

$$\frac{\partial}{\partial T} p_x(t,T) = -\int_0^x \left(m_{z,x}(t,T) \frac{\partial}{\partial z} p_z(t,T) \right) dz, \tag{31}$$

where $p_x(t,T)$ are, as always, defined by (4).

Proof. The constraint (30) follows form the fact that $l(\cdot)$ is restricted to $[0,1]$. To prove (31), we have

$$\begin{aligned}
\frac{\partial}{\partial T} p_x(t,T) &= -\mathsf{P}\left(l(T) \le x, l(T+dT) > x \middle| \mathscr{M}_t\right)/dT \\
&= -\int_0^x \mathsf{P}\left(l(T) \in dz, l(T+dT) > x \middle| \mathscr{M}_t\right)/dT \\
&= -\int_0^x \mathsf{P}\left(l(T+dT) > x \middle| l(T) = z, \mathscr{M}_t\right) \mathsf{P}\left(l(T) \in dz \middle| \mathscr{M}_t\right)/dT \\
&= -\int_0^x m_{z,x}(t,T) \frac{\partial}{\partial z} p_z(t,T)\, dz
\end{aligned}$$

as stated.∎

Remark 4.3. *If $p_0(t,T) > 0$, then the density $\frac{\partial}{\partial x} p_x(t,T)$ will be unbounded as $x \to 0^+$ and the integral in (31) is defined as the limit value.*

Definition 4.2. *The jump survival function $m_{z,x}(t,T)$ is said to be* consistent *with the set of loss probabilities if (31) is satisfied.*

Given the loss probabilities $p_x(t,T)$, the jump survival functions (and, thus, the actual dynamics of the loss process itself) can be found by solving the equation (31) for $m_{z,x}(t,T)$. A general and sound procedure for solving such a severely under-determined problem as finding a process consistent with one-dimensional marginal distributions is based on the idea of relative entropy minimization. Suppose a process $\tilde{l}(\cdot)$ is chosen from financial considerations as the "ideal" loss process. Denote by $\mathscr{I}(l,\tilde{l})$ the relative entropy of $l(\cdot)$ with respect to $\tilde{l}(\cdot)$ (without going into much detail, relative entropy measures how far the "candidate" process $l(\cdot)$ is from the "ideal" $\tilde{l}(\cdot)$). Then, $l^*(\cdot)$ is defined as a solution to

$$l^*(\cdot) = \arg\min\left\{\mathscr{I}(l,\tilde{l}), \text{ subject to (31)}\right\}.$$

The method finds the process closest to the ideal that satisfies the constraints on marginal distributions. Methods based on entropy minimization have proven to be quite powerful in related areas of quantitative finance, for example in fitting a process for a given asset to European option prices on that asset, see eg Avellaneda et al. (1997).

This line of enquiry is left for future work to explore. Here we focus on a somewhat simpler idea of suitably parametrizing the jump survival function $m_{z,x}(t,T)$ and choosing the parameters so that (31) is satisfied. The parametrization should be suitably parsimonious so that fitting to (31) will uniquely define the process (within the parametric class).

A spatially-homogeneous (conditional) Markov chain would have the jump survival function $m_{z,x}(t,T)$ dependent on the difference $x-z$ only, $m_{z,x}(t,T) = \theta(T, x-z)$. Such a process, however, cannot satisfy (30) and needs to be suitably modified. This observation motivates the following form,

$$m_{z,x}(t,T) = \theta(T, x-z)\, v_x(t,T), \tag{32}$$

where the function $\theta(T,y) \ge 0$ is given externally, and $v_x(t,T)$ is chosen so that (31) is satisfied. The function $\theta(\cdot,y)$ should have a hump around the value of y that we identify as the most likely per name loss-given-default. The following theorem shows how the suitable normalizing function $v_x(t,T)$ can be found, and a loss process constructed.

Theorem 4.2. *Suppose the conditions (5) hold, a time $t > 0$ is fixed, and the spatially-homogeneous component of the jump survival function, $\theta(T,y) \geq 0$ is given such that*

$$\int_0^x \left(\theta(T,x-z) \frac{\partial}{\partial z} p_z(t,T) \right) dz > 0 \text{ for any } t,T,x.$$

Then there exists a non-decreasing loss process $l(\cdot)$, on the state space $[0,1]$, $l(0) = 0$, such that

$$P(l(T) \leq x | \mathscr{M}_t) = p_x(t,T)$$

for any $x \in [0,1]$, $0 \leq t \leq T$, and such that

$$\mathsf{P}(l(T+dT) > x | l(T) = z, \mathscr{M}_t) = m_{z,x}(t,T)\, dT,$$

for $m_{z,x}(t,T)$ of the form

$$m_{z,x}(t,T) = \theta(x-z,T) v_x(t,T),$$

where

$$v_x(t,T) = \frac{-\frac{\partial}{\partial T} p_x(t,T)}{\int_0^x \left(\theta(T,x-z) \frac{\partial}{\partial z} p_z(t,T) \right) dz} \geq 0.$$

In particular, $m_{z,x}(t,T)$ is consistent with the loss probabilities in the sense of Definition 4.2.

Proof. Substituting (32) into the consistency condition (31), we obtain

$$\frac{\partial}{\partial T} p_x(t,T) = - \int_0^x \left(\theta(t,x-z) v_x(t,x) \frac{\partial}{\partial z} p_z(t,T) \right) dz.$$

Solving for $v_x(t,T)$ we obtain

$$v_x(t,T) = \frac{-\frac{\partial}{\partial T} p_x(t,T)}{\int_0^x \left(\theta(T,x-z) \frac{\partial}{\partial z} p_z(t,T) \right) dz}.$$

The denominator is positive by assumption, hence $v_x(t,T)$ is non-negative, and so is $m_{z,x}(t,T)$ defined by (32). Thus, a process $l(\cdot)$ can be defined as a conditional Markov process with jump survival function $m_{z,x}(t,T)$. Finally, such $m_{z,x}(t,T)$ automatically satisfies (30) because $p_1(t,T) \equiv 1$.
∎

Remark 4.4. *Choosing*

$$\theta(T,y) = 1_{\{y \in [0,1/N]\}},$$

a simple one-step Markov chain on the grid $\{j/N\}_{j=0}^N$, as constructed in Section 4.1, is recovered.

Remark 4.5. *All remarks to Theorem 4.1 apply to this theorem as well.*

5. VALUATION

5.1. **Tranche spreads and other market observables.** A tranche CDO swap (tranche swap) with maturity T, attachment point a, and detachment point d is defined as a contract that over the period $[0,T]$ pays all portfolio default losses in the interval $[a,d]$, in exchange for a payment of a fixed rate applied to a loss-depreciating notional. To formally define this instrument, introduce a tenor structure

$$\begin{aligned} &T_0 < T_1 < \cdots < T_N, \\ &\Delta_n = T_n - T_{n-1}, \end{aligned}$$

and define "call spread" payouts $C(t) = \max(l(t)-a,0) - \max(l(t)-d,0)$. Then formally (and simplifying the definition somewhat in the interest of clarity), the value of a tranche swap[7] with fixed rate k at time $t < T_0$ is given by

$$\begin{aligned} V_{\text{swap}}(t) = &\sum_{n=1}^{N} P(t,T_n)\,\mathsf{E}\left(C(T_n) - C(T_{n-1}) \mid \mathscr{L}_t\right) \\ &- k\sum_{n=1}^{N} \Delta_n P(t,T_n)\,\mathsf{E}\left(d - a - C(T_n) \mid \mathscr{L}_t\right), \end{aligned} \tag{33}$$

where $P(t,T)$ denotes the time t value of a riskless zero-coupon bond with maturity T.

The break-even tranche spread at time t is defined as the fixed rate k that makes $V_{\text{swap}}(t)$ have zero value. Denoting it by $R(t)$, we get

$$R(t) = \frac{\sum_{n=1}^{N} P(t,T_n)\,\mathsf{E}\left(C(T_n) - C(T_{n-1}) \mid \mathscr{L}_t\right)}{D(t)},$$

where the *tranche PVBP* $D(t)$ is given by

$$D(t) = \sum_{n=1}^{N} \Delta_n P(t,T_n)\,\mathsf{E}\left(d - a - C(T_n) \mid \mathscr{L}_t\right).$$

Recall that the loss filtration $\mathscr{L}_t$ consists of the information from $\mathscr{M}_t$ (the background filtration) and the loss process information up to time t. Since by construction the loss process is Markovian conditioned on the background filtration $\mathscr{M}_t$, we can write

$$\begin{aligned} \mathsf{E}\left(X \mid \mathscr{L}_t\right) &= \mathsf{E}\left(X \mid \mathscr{M}_t, l(t)\right) \\ &= \mathsf{E}\left(X \mid \{p_\cdot(s,\cdot)\}_{s=0}^{t}, l(t)\right), \end{aligned}$$

for any suitable random variable X, where $\{p_\cdot(s,\cdot)\}_{s=0}^{t}$ are the paths of $p_x(s,T)$ for $s \in [0,t]$ for all x, T. Importantly, to compute $V_{\text{swap}}(t)$, $R(t)$ and $D(t)$ at time t as functions of the state variables at time t (p_x's and $l(t)$), we need to be able to compute

$$\mathsf{E}\left(f(l(T)) \mid \mathscr{M}_t, l(t)\right)$$

for various functions $f(\cdot)$ and time horizons T. These can be decomposed into linear combinations of

$$p_{y,x}(t,T) := \mathsf{P}\left(l(T) \le x \mid \mathscr{M}_t, l(t) = y\right).$$

---

[7] We note that since $l(t)$ represents normalized portfolio loss (rather than actual loss $A(0) - A(t)$), the stated payouts here are normalized as well.

Note that these are different from $p_x(t,T)$, the primitives of the model. In fact, the following holds,

$$\begin{aligned} p_x(t,T) &= \int_0^1 p_{y,x}(t,T)\,\mathsf{P}\left(l(t)\in dy \middle| \mathscr{M}_t\right) \\ &= \int_0^1 p_{y,x}(t,T)\,\pi_y(0,t)\,dy, \end{aligned}$$

where π is defined by (7). In essence, the primitives of the model $p_x(t,T)$ provide information on the *average* of default loss probabilities (averaged over the number of losses to the observation date t), whereas the quantities often required are $p_{y,x}(t,T)$, the loss probabilities *conditioned on a particular loss level* at observation time t. In the next section we discuss how to compute these quantities, the building blocks for securities valuation.

Remark 5.1. *At time 0, $p_{l(0),x}(0,T)=p_{0,x}(0,T)=p_x(0,T)$, and $\mathsf{E}\left(f(l(T))\middle|\mathscr{M}_0, l(0)\right)=\mathsf{E}\left(f(l(T))\middle|\mathscr{M}_0\right)$ can be computed by direct numerical integration over the loss distribution $\pi_x(0,T)=\partial p_x(0,T)/\partial x$. If tranche swaps covering intervals $[0,d]$ trade at many levels of d, this in turn will allow us to recover $p_x(0,T)$ from market prices of tranche swaps.*

5.2. **Computing portfolio loss probabilities conditioned on the level of loss at observation time.** While the conditional probabilities $p_{x,y}(t,T)$ are not directly observable from the background filtration, they can be efficiently computed by solving the appropriate forward Kolmogorov equations, as the following proposition demonstrates.

Proposition 5.1. *Fix $t\geq 0$ and $y\in[0,1]$, and let the jump survival functions $m_{x,z}(t,T)$, consistent with $p_x(t,T)$ (in the sense of Definition 4.2) be given for all $x,z\in[0,1]$ and for all $T\geq t$ (for example, the ones obtained in Theorem 4.1 or Theorem 4.2 could be used). Then all conditional probabilities $\left\{p_{x,y}(t,T)\right\}_{x,T}$ can be obtained by solving a system of forward Kolmogorov equations in T and x,*

$$\frac{\partial}{\partial T}p_{x,y}(t,T) = -\int_0^x \left(m_{z,x}(t,T)\frac{\partial}{\partial z}p_{z,y}(t,T)\right)dz, \quad T\geq t, \quad x\in[0,1], \tag{34}$$

with the initial conditions at time $T=t$,

$$p_{x,y}(t,T=t) = 1_{\{x\geq y\}}. \tag{35}$$

Proof. The proof closely mirrors that of Proposition 4.1, with a small twist. We have,

$$\begin{aligned} \frac{\partial}{\partial T}p_{x,y}(t,T) &= -\mathsf{P}\left(l(T)\leq x, l(T+dT)>x\middle|\mathscr{M}_t, l(t)=y\right)/dT \\ &= -\int_0^x \mathsf{P}\left(l(T)\in dz, l(T+dT)>x\middle|\mathscr{M}_t, l(t)=y\right)/dT \\ &= -\int_0^x \mathsf{P}\left(l(T+dT)>x\middle|l(T)=z,\mathscr{M}_t, l(t)=y\right)\mathsf{P}\left(l(T)\in dz\middle|\mathscr{M}_t, l(t)=y\right)/dT \\ &= -\int_0^x \mathsf{P}\left(l(T+dT)>x\middle|l(T)=z,\mathscr{M}_t, l(t)=y\right)\frac{\partial}{\partial z}p_{z,y}(t,T)\,dz/dT. \end{aligned}$$

Since $\mathscr{M}_t \subset \mathscr{M}_T$, we have

$$\mathsf{P}\left(l\left(T+dT\right)>x\middle|\, l\left(T\right)=z,\mathscr{M}_t,l\left(t\right)=y\right)=\mathsf{E}\left(\mathsf{E}\left(1_{\{l(T+dT)>x\}}\middle|\, l\left(T\right)=z,\mathscr{M}_T,l\left(t\right)=y\right)\middle|\,\mathscr{M}_t\right).$$

Conditional on $\mathscr{M}_T$, the process $\{l\left(s\right)\}_{s\leq T}$ is a Markov process, hence in the inner conditional expectation the conditioning on $l\left(t\right)=y$ can be dropped,

$$\mathsf{E}\left(1_{\{l(T+dT)>x\}}\middle|\, l\left(T\right)=z,\mathscr{M}_T,l\left(t\right)=y\right)=\mathsf{E}\left(1_{\{l(T+dT)>x\}}\middle|\, l\left(T\right)=z,\mathscr{M}_T\right).$$

Again using the fact that $\mathscr{M}_t \subset \mathscr{M}_T$, we obtain

$$\mathsf{E}\left(\mathsf{E}\left(1_{\{l(T+dT)>x\}}\middle|\, l\left(T\right)=z,\mathscr{M}_T\right)\middle|\,\mathscr{M}_t\right)=\mathsf{E}\left(1_{\{l(T+dT)>x\}}\middle|\, l\left(T\right)=z,\mathscr{M}_t\right)$$

and finally

$$\begin{aligned}\mathsf{P}\left(l\left(T+dT\right)>x\middle|\, l\left(T\right)=z,\mathscr{M}_t,l\left(t\right)=y\right) &= \mathsf{P}\left(l\left(T+dT\right)>x\middle|\, l\left(T\right)=z,\mathscr{M}_t\right)\\ &= m_{z,x}\left(t,T\right)dT.\end{aligned}$$

Thus,

$$\frac{\partial}{\partial T}p_{x,y}\left(t,T\right)=-\int_0^x m_{z,x}\left(t,T\right)\frac{\partial}{\partial z}p_{z,y}\left(t,T\right)dz$$

and (34) is proved. The initial conditions (35) follow trivially.

∎

Remark 5.2. *The equations (34) are the same as satisfied by the loss probabilities not conditioned on the loss level,* $p_x\left(t,T\right)$*; the difference is only in the initial conditions (35).*

Remark 5.3. *When discretized in* $x\in[0,1]$*, the equations (34) become a system of ODEs that are trivial to solve numerically.*

Remark 5.4. *Let us emphasize that the algorithm above allows us to compute* $p_{y,x}\left(t,T\right)$ for all x *at the same time. This is very important as calculations of*

$$\mathsf{E}\left(f\left(l\left(T\right)\right)\middle|\,\mathscr{M}_t,l\left(t\right)\right)$$

(the motivation behind our analysis) by replication require $p_{l(t),x}\left(t,T\right)$ for all x. *Moreover, if* $\mathsf{E}\left(f\left(l\left(T\right)\right)\middle|\,\mathscr{M}_t,l\left(t\right)\right)$ *for various* T *are required in the same simulation (as is the case if* $V_{\text{swap}}\left(t\right)$ *in (33) is to be evaluated), then* only one *forward Kolmogorov equation needs to be solved to the final* T.

5.3. **Examples of securities and their valuation.**

5.3.1. *Option on a tranche.* Recall the definition of the tranche swap in (33). Consider a tranche swaption, ie, an option to enter the swap at time $T=T_0$, at the break-even spread k agreed at time 0, on a tranche swap with the attachment-detachment interval $[a,d]$ (also agreed at time 0). The value of such security is given by

$$V_{\text{swaption}}\left(0\right)=P\left(0,T\right)\mathsf{E}\left(V_{\text{swap}}\left(T\right)\right)^{+},$$

where

$$V_{\text{swap}}\left(T\right)=\mathsf{E}\left(\sum_{n=0}^{N}f_n\left(l\left(T_n\right)\right)\middle|\,\mathscr{L}_T\right). \tag{36}$$

and the functions $f_n\left(\cdot\right)$ are easily obtained form the representation (33).

In view of Proposition 5.1 and Remark 5.4, the tranche swaption can be priced using the following algorithm.

(1) Simulate a path of the background process $\{p_x(u,s)\}$ for $u \in [0,T]$ for all x and s using (14);
(2) Simulate a path of the loss process $l(u)$ for $u \in [0,T]$;
(3) Compute $\mathsf{E}(f_n(l(T_n))|\mathscr{M}_T, l(T))$ for all $n = 0,\ldots,N$ by solving forward Kolmogorov equations from Proposition 5.1 and Remark 5.4, using $y = l(T)$ in the initial conditions;
(4) Compute the payoff $\left(\sum_{n=0}^{N} \mathsf{E}(f_n(l(T_n))|\mathscr{M}_T, l(T))\right)^+$ for the simulated path;
(5) Average over all paths and discount.

Valuation of many other securities follows the same pattern. Bermuda-style options on tranche swaps can be handled by the now-well-established methods of valuing American-style securities in Monte-Carlo. The Longstaff-Schwartz algorithm (see eg Longstaff and Schwartz (1998) or, more relevantly, Piterbarg (2005)), or the Andersen algorithm (see Andersen (1998)) is usually required to estimate continuation values at exercise dates.

5.3.2. *Leveraged super-senior tranche.* Leveraged super-senior tranches with loss triggers (LSS-LT henceforth) are interesting and challenging instruments, involving an "exotic" option payout on the joint distribution of losses and portfolio credit spreads. In an LSS-LT contract, an investor enters into a senior tranche swap with attachment-detachment interval $[a,d]$, yet is liable only for a small posted collateral $U < d - a$. To partially protect the protection buyer against default losses that will exhaust the entire collateral, the contract is settled on mark-to-market terms the first time the portfolio loss process $l(t)$ exceeds a pre-specified trigger level. Let $K(t)$ denote this trigger level (typically $K(t)$ is an increasing deterministic function of time) and define

$$\tau = \inf\{t : l(t) \geq K(t)\}.$$

Provided that τ is less than deal maturity T, the mark-to-market mechanism at time τ will pay the protection buyer

$$\min\left(U, V_{\text{swap}}(\tau)\right)$$

where V_{swap} is the value of the underlying tranche CDO swap. From the perspective of the protection buyer, the LSS-LT swap is thus equal to a long position in a standard CDO swap combined with a short position in a path-dependent "trigger" option that pays

$$V_{\text{trig}}(\tau) = \left(U - V_{\text{swap}}(\tau)\right)^+ \tag{37}$$

at time τ, provided that $\tau < T$.

To price an LSS-LT swap in the SPA model framework, it suffices to price the short position in the trigger option with payout (37). Using the notation introduced in equation (36), a basic pricing is listed below.

(1) Simulate a path of the background process $\{p_x(u,s)\}$ for $u \in [0,T]$ for all x and s using (14);
(2) Simulate a path of the loss process $l(u)$ for $u \in [0, \min(T,\tau)]$;
(3) Compute $1_{\{\tau<T\}} \cdot \mathsf{E}(f_n(l(T_n))|\mathscr{M}_\tau, l(\tau))$ for all $n = 0,\ldots,N$ by solving forward Kolmogorov equations from Proposition 5.1 and Remark 5.4, using $y = l(\tau)$ in the initial conditions;

(4) Compute the payoff $1_{\{\tau<T\}} \cdot P(0,\tau)\left(U - \sum_{n=0}^{N} \mathsf{E}\left(f_n\left(l\left(T_n\right)\right) \middle| \mathscr{M}_\tau, l\left(\tau\right)\right)\right)^+$ for the simulated path;
(5) Average over all paths.

The algorithm is closely related to the basic algorithm for a tranche option (see Section 5.3.1), with the added twist of option exercise taking place at the random trigger time τ.

6. MODEL EXAMPLES

So far we have concentrated on developing a fairly general framework for valuing derivatives on the dynamics of the loss distribution. In this section, we endeavor to present concrete examples of the models in the SPA framework. For simplicity, we focus on one-dimensional dynamics of the loss probabilities p_x (or forward rates f_x). Given the wealth of knowledge available on HJM models of interest rates, it is a sensible approach to adapt an HJM model to propagate multiple term curves indexed by x at the same time, while preserving the spatial ordering condition (5c) (or equivalent). Note that, once the (consistent) dynamics of the loss distribution are specified, the loss process itself is easily defined as in Section 4.1 or Section 4.2. Hence, we concentrate on the loss distribution dynamics only.

6.1. **SPA model of loss probabilities.**

Our first example is inspired by the sufficient conditions of Theorem 3.1. Consider a model where the diffusion of each loss probability process depends on that loss probability only. In particular, let us choose a sufficiently smooth deterministic function $\phi\left(t,T,p\right)$ (satisfying the usual growth conditions) and define

$$dp_x\left(t,T\right)/p_x\left(t,T\right) = \phi\left(t,T,p_x\left(t,T\right)\right) dW\left(t\right), \quad 0 \le x \le 1, \quad 0 \le t,T \le T_{\max}.$$

We require

$$\begin{aligned} \phi\left(T,T,p\right) &\equiv 0, \\ \phi\left(t,T,1\right) &\equiv 0. \end{aligned}$$

If $\phi\left(t,T,p\right)$ is differentiable everywhere, the conditions of Theorem 3.1 are satisfied. For example, we may use

$$\phi\left(t,T,p\right) = \left(1-p\right)^b \frac{1-e^{-a(T-t)}}{a}, \quad b \ge 1/2.$$

Then the volatility structure can be controlled in the spatial (by changing b) and tenor (by changing a) dimensions.

This model is simple but Monte-Carlo simulation can be computationally intensive. If the state dimension (x) is discretized with N points and the tenor dimension (T) with M points, then the total of NM variables will need to be propagated. Also, specifying the volatility structure in terms of the loss probability dynamics may be somewhat unintuitive. The next example addresses this issue.

6.2. **SPA model of instantaneous forward rates.** Assume that the initial structure of instantaneous forward rates satisfies

$$x \geq y \Rightarrow f_x(0,T) \leq f_y(0,T) \qquad \text{for } 0 \leq x, y \leq 1, \quad 0 \leq T \leq T_{\max}.$$

Then a model can be built using the sufficient conditions of Theorem 3.2. In particular, we choose a sufficiently smooth deterministic function $\varphi(t,T,f)$ (satisfying the usual growth conditions) and define

$$d f_x(t,T) = \varphi(t,T,f_x(t,T)) \left(\int_t^T \varphi(t,u,f_x(t,u))\, du \right) dt + \varphi(t,T,f_x(t,T))\, dW(t),$$
$$0 \leq x \leq 1, \quad 0 \leq t, T \leq T_{\max}.$$

In order to satisfy the conditions of Theorem 3.2, we require that

$$\begin{aligned} \varphi(t,T,f) &\geq 0, \\ \varphi(t,T,0) &\equiv 0, \end{aligned}$$

and

$$f_1 \leq f_2 \Rightarrow \varphi(t,T,f_1) \leq \varphi(t,T,f_2).$$

This is sufficient to have a consistent model. The volatility structure is more intuitively specified in terms of forward rates. For example, we may use

$$\varphi(t,T,f) = e^{-a(T-t)} f^b, \quad 0 < b < 1.$$

Then the volatility structure can be controlled in the spatial (by changing b) and tenor (by changing a) dimensions.

Just like the model in the previous section, it also suffers from the high dimensionality issue.

6.3. **Forward Libor SPA model.** To construct a forward Libor SPA model, the results of Section 3.3 can be used. Assume that the initial structure of forward Libor rates (see definition (24)) satisfies (28). To obtain a consistent SPA model, a local volatility function $\psi(v)$ satisfying (29) needs to be chosen. The *log-normal forward Libor SPA model* is specified by choosing

$$\psi(x) = x.$$

It is trivial to check that (29) is satisfied, and thus the model is consistent. Staying in the context of one-factor models, we choose the one-dimensional volatility structure $\{\gamma(t,n)\}_n$ and define

$$\begin{aligned} dF_x(t,n)/F_x(t,n) &= \gamma(t,n)\left(dW(t) + \mu_x(t,n)\, dt\right), \\ \mu_x(t,n) &= \sum_{k=n(t)}^{n} \frac{\Delta_k F_x(t,k)}{1+\Delta_k F_x(t,k)} \gamma(t,k). \end{aligned}$$

As with forward Libor model, Monte-Carlo simulation involves simultaneous propagation of the forward Libor rates $F_x(\cdot,n)$ for all x and n.

6.4. **Gaussian SPA model.** The previous three examples demonstrated that models of consistent loss dynamics can be found relatively easily, but they tend to be high-dimensional and thus potentially computationally expensive to run. Similar challenges in interest rate modelling have been addressed by developing Markov models of interest rates, where the whole term curve at any point in time can be reconstructed from a small number of Markovian factors. Let us explore this avenue for the SPA framework.

The simplest one-factor Markovian HJM model (in fact, the only one) is the Markov Gaussian model, with forward rates following Gaussian processes. The Gaussian assumption, of course, immediately violates the requirement that the forward rates should be positive at all times (a consequence of (22b)). While clearly not ideal, we note that the same problem has not been found particularly relevant when valuing interest rate exotics, and we proceed nonetheless.

Accepting negative forward rates in the SPA model requires relaxing the assumption that the loss process is non-decreasing. In fact, if for a particular s and x, $f_x(t,s)$ is negative, it means that the loss process has to be able to *decrease* in that state of the world. This corresponds to a certain windfall in the underlying portfolio, a feature that is not particularly realistic, but is likely to have only limited impact on the valuation of portfolio loss derivatives.

The Gaussian model of interest rates is parametrized by the volatility σ and mean-reversion a of the short rate. To define a model, assume that there is a volatility and mean reversion $\sigma(x)$ and $a(x)$ given for each loss level x. We define the *Gaussian one-factor SPA model* by imposing the following volatility structure on the forward rates,

$$\begin{aligned}\sigma_x(t,T) &= \sigma(x)\, e^{-a(x)(T-t)},\\ \Sigma_x(t,T) &= -\sigma(x)\,\frac{1-e^{-a(x)(T-t)}}{a(x)},\end{aligned}$$

$$d f_x(t,T) = \Sigma_x(t,T)\,\sigma_x(t,T)\,dt + \sigma_x(t,T)\,dW(t)\,.$$

We note that for each x, the collection of forward rates $f_x(t,\cdot)$ can be expressed in terms of the short rate $\lambda_x(t)$,

$$\begin{aligned}f_x(t,T) &= f_x(0,T) + e^{-a(x)(T-t)}\left(\lambda_x(t) - f_x(0,t) + b(T-t)\,\theta_x(t)\right),\\ b(\tau) &= \frac{1-e^{-a\tau}}{a}.\end{aligned}$$

where the short rate follows

$$d\lambda_x(t) = (\theta_x(t) - a(x)\,\lambda_x(t))\,dt + \sigma(x)\,dW(t)\,,\qquad \lambda_x(0) = 0,$$

and the deterministic function $\theta_x(t)$ is suitably defined.

It is quite clear that (5c) can only be satisfied if $\sigma(x)$, $a(x)$ do not depend on x,

$$\sigma(x) \equiv \sigma,\qquad a(x) \equiv a.$$

This is a general feature of a model in the SPA framework – diffusion coefficients can depend on the spatial variable x only through their dependence on the dynamic variables (eg f_x, p_x) and *not explicitly*.

Note that this unfortunately violates another requirement on $\Sigma_x(t,T)$, namely (13).

6.5. **Quasi-Gaussian SPA model.** In view of the fact that an attempt to construct a Gaussian SPA model in the previous section results in a model that violated two out of the three conditions in (5), another attempt at building a simple, preferably Markovian, SPA model is warranted.

In interest rate modelling, a Markov Gaussian model can be extended to incorporate a local volatility function at the expense of having another state variable; such models are often called *quasi-*, or *pseudo-Gaussian* (see Jamshidian (1991), Babbs (1990)). Let us apply this idea to modelling the propagation of the loss distribution.

Assume that a local volatility function $\sigma(t,v)$, a function of time t and state v, is specified. Also assume a mean reversion parameter a is given. The model is specified in terms of the "short rate" processes $\lambda_x(t) = f_x(t,t)$. Thus we define, for each $x \in [0,1]$,

$$\begin{aligned} d\lambda_x(t) &= (\theta_x(t) - a\lambda_x(t))\,dt + \sigma(t,\lambda_x(t))\,dW(t), \\ \theta_x(t) &= \frac{\partial}{\partial t} f_x(0,t) + a f_x(0,t) + \int_0^t e^{-2a(t-s)}\sigma^2(s,\lambda_x(s))\,ds. \end{aligned} \tag{38}$$

Note that $\theta_x(t)$ is stochastic, through its dependence on the path of $\lambda_x(s)$. Each forward rate can be represented in terms of the two state variables $\lambda_x(t)$ and $\theta_x(t)$,

$$f_x(t,T) = f_x(0,T) + e^{-a(T-t)}\left[\lambda_x(t) - f_x(0,t) + b(T-t)\,\theta_x(t)\right], \tag{39}$$

where

$$b(\tau) = \frac{1-e^{-a\tau}}{a}.$$

Likewise, a loss probability for any expiry T can be represented in terms of the two state variables,

$$p_x(t,T) = \frac{p_x(0,T)}{p_x(0,t)} \exp\left(-b(T-t)\left[\lambda_x(t) - f_x(0,t) + \frac{1}{2}b(T-t)\,\theta_x(t)\right]\right). \tag{40}$$

If N loss levels $\{x_i\}$ are used in simulation, $2N$ state variables need to be propagated. This is a significant reduction of computational effort compared to the general, non-Markovian models from Sections 6.1 and 6.2.

The model is specified in terms of forward rates, and we need to impose the conditions (22a), (22b) and (23) on the initial term structure of forward rates ($t = 0$).

To satisfy the conditions (22a) and (22b), it is sufficient (and necessary) to require that $\sigma(t,0) \equiv 0$. In addition, just as in Theorem 3.2, the volatility structure needs to be non-decreasing in the space variable. Under these conditions the model can be shown to satisfy the consistency conditions, as the following proposition demonstrates.

Proposition 6.1. *Assume the initial forward rates satisfy (22a), (22b) and (23). Given the volatility function $\sigma(s,v)$ that satisfies*

$$\sigma(t,0) \equiv 0, \tag{41}$$

$$v_1 \geq v_2 \Rightarrow \sigma(t,v_1) \geq \sigma(t,v_2), \tag{42}$$

and the usual growth and continuity conditions. Also let the mean reversion parameter a satisfy

$$a \geq \max\left\{-\frac{\partial}{\partial s}\log\left(f_y(0,s) - f_x(0,s)\right), \quad \text{for all } s \in [0, T_{\max}],\ 0 \leq y < x \leq 1\right\}. \tag{43}$$

Then the model (38), (39), (40) satisfies the consistency conditions (22).

Proof. Both conditions (22a), (22b) follow from (41).

Fix $0 \leq y < x \leq 1$. By assumption $\lambda_x(0) \leq \lambda_y(0)$. Let us show that $\lambda_x(s) \leq \lambda_y(s)$ for any s. Let t be the first time when $\lambda_x(t) = \lambda_y(t)$. Then the difference $\lambda_y(\cdot) - \lambda_x(\cdot)$ will have zero diffusion and the drift

$$d_1 + d_2,$$

where

$$\begin{aligned} d_1 &= \left(\frac{\partial}{\partial t} f_y(0,t) - \frac{\partial}{\partial t} f_x(0,t)\right) + a\left(f_y(0,t) - f_x(0,t)\right), \\ d_2 &= \int_0^t e^{-2a(t-s)}\left(\sigma^2(s,\lambda_y(s)) - \sigma^2(s,\lambda_x(s))\right) ds. \end{aligned}$$

Clearly (43) implies that $d_1 \geq 0$, and (42) (together with the fact that $\lambda_y(s) \geq \lambda_x(s)$ for $s \leq t$) implies that $d_2 \geq 0$. Hence $\lambda_y(\cdot) - \lambda_x(\cdot)$ always has a positive drift at zero. Since the difference is non-negative to start with, it stays nonnegative at any time,

$$\lambda_y(t) \geq \lambda_x(t) \text{ for any } t \geq 0. \tag{44}$$

Next we note that (42) and (43) imply that

$$\theta_y(t) \geq \theta_x(t) \text{ for any } t \geq 0. \tag{45}$$

Now let us show that $f_x(t,T) \leq f_y(t,T)$ for any $T \geq t$ and $y \leq x$. Integrating (43) from t to T, we obtain

$$a(T-t) \geq -\left(\log\left(f_y(0,T) - f_x(0,T)\right) - \log\left(f_y(0,t) - f_x(0,t)\right)\right).$$

Taking the exponentials,

$$e^{-a(T-t)} \leq \frac{f_y(0,T) - f_x(0,T)}{f_y(0,t) - f_x(0,t)},$$

which implies

$$f_y(0,T) - f_x(0,T) \geq e^{-a(T-t)}\left(f_y(0,t) - f_x(0,t)\right). \tag{46}$$

Recalling (39),

$$\begin{aligned} f_y(t,T) - f_x(t,T) &= d_3 + d_4 + d_5, \\ d_3 &= \left(f_y(0,T) - f_x(0,T)\right) - e^{-a(T-t)}\left(f_y(0,t) - f_x(0,t)\right), \\ d_4 &= e^{-a(T-t)}\left(\lambda_y(t) - \lambda_x(t)\right), \\ d_5 &= b(T-t)\left(\theta_y(t) - \theta_x(t)\right). \end{aligned}$$

Then $d_3 \geq 0$ by (46), $d_4 \geq 0$ by (44), $d_5 \geq 0$ by (45).

■

7. CONCLUSIONS

In this paper we have presented the SPA model, a framework for the dynamics of portfolio losses. The model is automatically calibrated to the initial implied marginal loss distributions and its dynamics—defined in terms of forward rates—propagate these initial loss distributions in time. In order to preserve the interpretation in terms of probability distributions the dynamics must satisfy certain constraints. We gave sufficient conditions on the joint volatility structure of the loss probabilities that guarantee the constraints in the HJM-like, as well as the BGM-like, settings.

For a good understanding of the model in general and in particular for pricing of derivatives on portfolio losses, it is necessary to know the stochastic process for losses. We gave a number of specific constructions for a loss process consistent with the simulated dynamics of loss probabilities (all in the class of conditional Markov processes), and discussed pros and cons of various choices. In the process we demonstrated that the *necessary* conditions for probability distributions are in fact *sufficient* to prevent arbitrage in the model (as the loss process can be explicitly constructed). We demonstrated that numerically efficient algorithms can be constructed to value securities linked to the dynamics of loss distributions, such as options on tranches. In particular, specific algorithms were presented for pricing an option on a tranche and a leveraged super-senior tranche, respectively.

This paper has focused on the formal development of the SPA model and the numerical routines needed to support it. Much work remains in empirical examination of the model behavior, as well as in the construction of robust dynamic parameter estimation, as required for practical trading applications of the model. The latter is obviously complicated by the quite limited set of liquidly traded portfolio instruments currently available, so it remains to be seen to what extent historical data can assist in the model construction. Also, for efficient implementation of volatility calibration to CDO options, is would be useful to develop fast approximative methods for pricing such options in the SPA setting, without the need to invoke Monte Carlo simulation. Several techniques from interest rate derivatives (see e.g. Andersen and Andreasen (2000) or Sidenius (2000)) may be useful for this.

Another interesting area of future research concerns hedging strategies in the SPA framework and, more generally, for models formulated in the portfolio loss space. For such models hedge computations are most naturally done in terms of perturbations to the loss distribution directly. This in turn will allow us to express hedges in terms of instruments that depend directly on the loss distribution, such as single-tranche CDOs. Hedges at the level of the individual firms are, by construction, difficult to obtain in the SPA model (and indeed in any model that works directly with portfolio losses), and some amount of ad-hoc reliance on a copula to link firm data (spreads and recovery rates) to the loss distribution may be necessary.

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APPENDIX A. PROOF OF THEOREM 3.1

Fix x,y such that $0 \leq x < y \leq 1$ and define

$$r_1(t) = p_y(t,T) - p_x(t,T).$$

Clearly $r_1(0) \geq 0$. We will demonstrate that if (19) holds, then $r_1(t) \geq 0$ P-a.s. for any $t \geq 0$, which will prove the theorem.

From (14)

$$\begin{aligned} dr_1(t) &= \sum_\alpha \left(p_y(t,T)\Sigma_y^\alpha(t,T) - p_x(t,T)\Sigma_x^\alpha(t,T)\right) dW^\alpha(t) \\ &= a(t)\, dZ(t), \end{aligned}$$

where

$$\begin{aligned} a^2(t) &= \sum_\alpha \left(p_y(t,T)\Sigma_y^\alpha(t,T) - p_x(t,T)\Sigma_x^\alpha(t,T)\right)^2, \\ dZ(t) &= a^{-1}(t)\sum_\alpha \left(p_y(t,T)\Sigma_y^\alpha(t,T) - p_x(t,T)\Sigma_x^\alpha(t,T)\right) dW^\alpha(t), \end{aligned} \tag{47}$$

and $Z(t)$ is a standard Brownian motion.

Let $f(r)$ be a twice-differentiable, bounded function, then by Ito's lemma

$$f(r_1(t)) = f(r_1(0)) + \frac{1}{2}\int_0^t a^2(s)\, f''(r_1(s))\, ds + \int_0^t f'(r_1(s))\, dZ(s).$$

Taking expected values and observing that the last term disappears, we obtain that

$$\begin{aligned} \mathrm{E} f(r_1(t)) &= f(r_1(0)) + \frac{1}{2}\int_0^t \mathrm{E}\left(a^2(t)\, f''(r_1(t))\right) ds \\ &= f(r_1(0)) + \frac{1}{2}\int_0^t \mathrm{E}\left(f''(r_1(t))\,\mathrm{E}\left(a^2(t)\middle| r_1(t)\right)\right) ds. \end{aligned}$$

Define $\varphi(x)$ by

$$\varphi^2(x) = \mathrm{E}\left(a^2(t)\middle| r_1(t) = x\right), \tag{48}$$

then

$$\mathrm{E} f(r_1(t)) = f(r_1(0)) + \frac{1}{2}\int_0^t \mathrm{E}\left(\varphi^2(r_1(s))\, f''(r_1(s))\right) ds. \tag{49}$$

Hence, for any (twice-differentiable bounded) function f,

$$\mathrm{E} f(r_1(t)) = \mathrm{E} f(r_2(t)),$$

where $r_2(\cdot)$ is given by

$$dr_2(t) = \varphi(r_2(t))\, dZ(t), \quad r_2(0) = r_1(0). \tag{50}$$

(Formally, this is so because $r_2(\cdot)$ solves the Martingale Problem (49), see Karatzas and Shreve (1991)).

By choosing $f(\cdot)$ to be a suitable approximation to $1_{\{x<0\}}$ we see that it is enough to prove that $r_2(t) \geq 0$ for any $t \geq 0$.

From (19), the definition (47) of $a(t)$, and the definition (48) of $\varphi(x)$ we obtain that

$$|\varphi(x)| \leq C^{1/2}\, |x|^{1/2}$$

for small x. Hence, the SDE (50) has a unique strong solution, see (Karatzas and Shreve, 1991, Chapter 5, Proposition 2.13). Since the solution with $r_2(0) = 0$ is $r_2(t) \equiv 0$, the nonnegativity of $r_2(\cdot)$ follows and the theorem is proved.

APPENDIX B. PROOF OF THEOREM 3.2

The only non-trivial condition to check is (23). Under the conditions of the theorem,

$$d f_x(t,T) = \sum_\alpha \varphi^\alpha(t,T,f_x(t,T)) \int_t^T \varphi^\alpha(t,u,f_x(t,u))\,du\,dt + \sum_\alpha \varphi^\alpha(t,T,f_x(t,T))\,dW^\alpha(t).$$

Note that the drift depends on stochastic quantities other than $f_x(t,T)$, so $f_x(\cdot,T)$ is not Markov and standard comparison theorems (such as, for example (Karatzas and Shreve, 1991, Chapter 5, Proposition 2.18)) do not apply. The idea of the proof can be extended, however.

Choose $0 \le y \le x \le 1$. By assumption, $f_x(0,T) \le f_y(0,T)$ for all T. Let τ be the first time for which the set $\Theta_\tau := \left\{T \in [\tau, T_{\max}] \,|\, f_x(\tau,T) = f_y(\tau,T)\right\}$ is non-empty. By continuity we have for $T \in \Theta_\tau$ that $f_x(\tau,u) \le f_y(\tau,u)$ for all $u \le T$. Consider now at time τ the process for $f_y(t,T) - f_x(t,T)$. Clearly, the diffusion term vanishes here and thus the process is pure drift

$$d(f_y - f_x)(t,T) = \sum_\alpha \varphi^\alpha(t,T,f_x(t,T)) \cdot \left(\int_\tau^T [\varphi^\alpha(t,u,f_y(t,u)) - \varphi^\alpha(t,u,f_x(t,u))]\,du\right) dt.$$

We observe that the drift coefficient is non-negative since $|\varphi^\alpha(t,u,v)|$ is non-decreasing in v for all α and all u. Hence then $f_x(t,T) \le f_y(t,T)$ for all t,T and the result follows.

APPENDIX C. CONTINUOUS-TIME LIMIT OF ONE-STEP MARKOV CHAINS

To avoid unnecessary complications, we assume that an unconditional Markov chain $l(t)$ consistent with a given set of one-dimensional marginal distributions

$$p(t,x) = \mathsf{P}(l(t) \le x), \tag{51}$$

is constructed per Section 4.1.

Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a discretization of the interval $[0,1]$. Then a discretized version of $l(t)$, that we denote by $l^N(t)$, can be constructed as a Markov chain on $\{x_j\}_{j=0}^N$ that can only jump one level up at each time t. If we denote the intensity of the transition $x_j \to x_{j+1}$ at time t by $\kappa^N(t,x_j)$, then it is shown previously that

$$\kappa^N(t,x_j) = \frac{-\partial p(t,x_j)/\partial t}{p(t,x_{j+1}) - p(t,x_j)}.$$

To construct a process consistent with $p(t,x)$ for all x, and given a collection of Markov chains $l^N(\cdot)$, it is tempting to take the limit $N \to \infty$ to obtain such a process. Considering this limit also can provide insights into the dynamics of the loss process approximation l^N for finite but large N.

Putting aside for a moment the question whether $\{l^N(\cdot)\}_N$ really converges weakly to anything (a question of relative compactness of the family of measures induced by these processes), let us just focus on identification of a limiting process, should one exist. For that, let us consider infinitesimal generators of $l^N(\cdot)$'s. If $f(x)$ is a test function (twice-differentiable and bounded), then the generator of l^N, denoted by $G^N(t)$, is given by

$$\left(G^N(t)f\right)(x_j)=\kappa^N(t,x_j)\left(f(x_{j+1})-f(x_j)\right).$$

Using the definition of $\kappa^N(t,x_j)$ we obtain

$$\begin{aligned}\left(G^N(t)f\right)(x_j) &= \frac{-\partial p(t,x_j)/\partial t}{p(t,x_{j+1})-p(t,x_j)}\left(f(x_{j+1})-f(x_j)\right)\\ &= -\frac{\partial p(t,x_j)}{\partial t}\times\frac{f(x_{j+1})-f(x_j)}{p(t,x_{j+1})-p(t,x_j)}\end{aligned}$$

and, assuming $\max_j \Delta x_j\to 0$ as $N\to\infty$, we obtain

$$\left(G^N(t)f\right)(x)\to(G(t)f)(x), \tag{52}$$

where

$$(G(t)f)(x)=\kappa(t,x)\times\frac{df(x)}{dx},\quad \kappa(t,x)=\frac{-\partial p(t,x)/\partial t}{\partial p(t,x)/\partial x}$$

for all x for which $\partial p(t,x)/\partial x$ exists.

Let $l(t)$ be the Markov process with the infinitesimal generator $G(t)$. The generator $G(t)$ can be easily recognized as implying non-stochastic dynamics of $l(t)$, ie

$$l(t)=l(s)+\int_s^t\kappa(t,l(u))\,du,\quad s<t. \tag{53}$$

At first glance this seems counter-intuitive – how can we get a deterministic process $l(t)$ consistent with (51)?

The resolution of this seeming paradox lies in the fact that the convergence result (52) does not hold for $t=0$, $x=0$, at which point $p(t,x)$ is non-differentiable. Hence, we must restrict our analysis to a subinterval $[\delta,T]$ for a suitable $\delta>0$. On this interval, the convergence holds. So, consider a set of random variables $\left\{l^N(\delta)\right\}_N$. Each of these has a distribution given by CDF $p(\delta,x)$. Hence, assume that the limiting process is only defined on $[\delta,T]$ with the initial value $l(\delta)$ being a random variable distributed according to $p(\delta,x)$. For $t>\delta$, the "dynamics" of the process are given by (53), ie deterministic. Let us construct such a "process" directly.

Let is denote

$$H(t,x)=p^{-1}(t,p(\delta,x)),\quad t\geq\delta,$$

where p^{-1} is the inverse of p in the second (ie x) argument. We note that

$$\begin{aligned}p(t,H(t,x)) &= x,\\ \frac{d}{dt}p(t,H(t,x)) &= 0,\\ \frac{\partial}{\partial t}p(t,H(t,x))+\frac{\partial}{\partial x}p(t,H(t,x))\times\frac{\partial}{\partial t}H(t,x) &= 0,\end{aligned}$$

so that

$$\frac{\partial}{\partial t}H(t,x)=\frac{-\frac{\partial}{\partial t}p(t,H(t,x))}{\frac{\partial}{\partial x}p(t,H(t,x))}. \tag{54}$$

Define $m(t) = H(t, l(\delta))$.Then

$$m(\delta) = l(\delta), \quad dm(t) = \frac{\partial}{\partial t} H(t, l(\delta))\, dt,$$

and, using (54),

$$dm(t) = \frac{-\frac{\partial}{\partial t} p(t, H(t, l(\delta)))}{\frac{\partial}{\partial x} p(t, H(t, l(\delta)))} dt = \kappa(t, m(t))\, dt.$$

Hence $m(t)$ satisfies the SDE (53) and thus necessarily

$$l(t) \equiv m(t), \quad t \geq \delta.$$

Let us directly verify that the process $l(t)$ thus constructed indeed satisfies (51). We have

$$l(t) = p^{-1}(t, p(\delta, l(\delta))). \tag{55}$$

We can recognize this as perhaps the simplest possible “mapping” of a random variable $l(\delta)$ with distribution $p(\delta, x)$ to a random variable $l(t)$ with distribution $p(t, x)$. Let us elaborate. The random variable $l(\delta)$ has CDF $p(\delta, x)$. Hence, the random variable $p(\delta, l(\delta))$ has uniform distribution. To obtain a random variable with the CDF $p(t, x)$, all we need to do is to apply the inverse of $p(t, x)$ to it, and that is exactly what happens in (55).

Clearly, the process $l(\cdot)$ obtained as a limit of one-step Markov chains has highly degenerate dynamics. Over the short period of time $[0, \delta]$, it “explodes” such that $l(\delta)$ has the right distribution $p(\delta, x)$; after that, it evolves deterministically to “reshape” the distribution function to conform to $p(t, x)$ at time t.

Since the limiting distribution of a sequence of Markov chains is highly dubious, the same can be said about the pre-limiting Markov chain for high N. In essence, a “burst of activity” happens over a short initial period of time, after which the process settles into a more-or-less deterministic pattern.

In conclusion we note that for a homogeneous portfolio of moderate size, the one-step Markov chain can serve as an adequate description of the loss process. If, however, the number of loss levels is large (either because LGDs are stochastic or highly inhomogeneous across names), the one-step Markov chain will likely give rise to a loss process of questionable dynamics.

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